Yiddish word of the day

"shreK" = שרעק

"fear" =

Yiddish phrase of the day

"A moshel iz nisht Keyne rai'ch" = אַ משל איז
נישט קיין ראַיה

"An example is not a proof" =

Lecture 3 – Linear Independence

Recall: Let $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_k \\ \downarrow & & \downarrow \end{pmatrix}_{n \times k}$ matrix.

Then we saw last class that the vectors $\vec{v}_1, \ldots, \vec{v}_k$ span $\mathbb{R}^n$ if and only if the matrix $\left(\begin{array}{ccc|c} \vec{v}_1 & \cdots & \vec{v}_k & \vec{b} \\ \downarrow & & \downarrow & \downarrow \end{array}\right)$ has a solution for every vector $\vec{b}$ in $\mathbb{R}^n$.

- **Note**: We don't require the solution to be unique.

Recall – Free Variables

- In an equation, the leading variable is
  - the first nonzero variable.

- A **free variable** in a system of equations is a variable that is **not** a leading variable in any of the equations.

ex)

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \longleftrightarrow$$

Leading variables: x_1, x_3

Free variable: x_2

Linear Independence

(want some "sense" of uniqueness of solutions)

Def: Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then we say $\vec{v}_1 \ldots \vec{v}_k$ are linearly independent if the only solution to the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \quad \text{is} \quad c_1 = c_2 = \cdots = c_k = 0$$

• if they are NOT LI we say they are linearly dependent.

2 questions

1) How to tell if any given vectors are linearly independent,

2) Who cares? What's the utility if they are?

ex) Determine if the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ are LI?

<u>Check</u>: Find all possible constants c_i such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$= \begin{pmatrix} c_1 \\ 2c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -c_3 \\ 3c_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 - 2c_2 - c_3 \\ 2c_1 + c_2 + 3c_3 \\ 0 + 0 + c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

these vectors <u>are LI!</u>

ex2: Are the vectors $\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, & \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$ LI?

Find $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & -1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

In this case c_4 is a free variable and the vectors are linearly dependent.

ex) Are the vectors $\overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{v_1}$, $\overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{v_2}$ and $\overbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}^{v_3}$ LI?

Again check:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c_3 is a free variable so they are not LI.

A pattern Emerges

- To answer Q1 - (How to tell) we do the following
  - Put our vectors as collumns of matrix
    - Put Matrix into EF
      - Check if there are free variables
        - If there are free variable
          - ⇒ they are NOT LI
        - If there are no free variables
          - ⇒ then they ARE LI!

- Thus if vectors are LI then their matrix has a pivot in every column!

• This tells us the following: If $\vec{v}_1 \dots \vec{v}_k$ are LI vectors in $\mathbb{R}^n$ then $k \leq n$

Q2 - Why do we care?

Suppose that $\vec{v}_1 \dots \vec{v}_k$ are LI vectors in $\mathbb{R}^n$

• Now let $\vec{b}$ in span$(\vec{v}_1, \dots, \vec{v}_k)$

(that is $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ for some $\overbrace{c_1 \dots c_k}^{\#\text{ in } \mathbb{R}}$)

• Then if $\vec{b} = d_1\vec{v}_1 + d_2\vec{v}_2 + \dots + d_k\vec{v}_k$ for some other $d_1 \dots d_k$

Then $c_1 = d_1, c_2 = d_2, \ldots c_k = d_k$

Proof: We have

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$$

$$\Rightarrow c_1\vec{v}_1 - d_1\vec{v}_1 + c_2\vec{v}_2 - d_2\vec{v}_2 + \cdots + c_k\vec{v}_k - d_k\vec{v}_k = \vec{0}$$

$$\Rightarrow (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_k - d_k)\vec{v}_k = \vec{0}$$

Since $\vec{v}_1, \ldots, \vec{v}_k$ are LI we have

$$0 = c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k$$

$$\Rightarrow c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k$$

Subspaces and Basis

Subspace - A subspace $W \subseteq \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ with the 2 properties

- If $\vec{v}_1, \vec{v}_2$ are in W then $\vec{v}_1 + \vec{v}_2$ is in W
- If c any real #, and $\vec{v}_1$ in W then $c\vec{v}$ is also in W

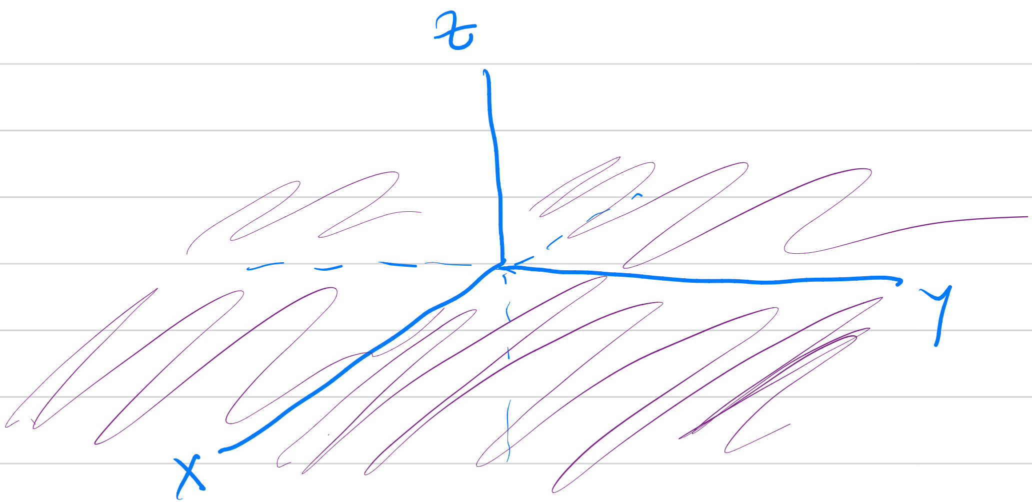
ex: Consider $\text{span}(\vec{v}_1 \ldots \vec{v}_k)$ for $\vec{v}_i$ in $\mathbb{R}^n$

Claim: $\text{Span}(\vec{v}_1 \ldots \vec{v}_k)$ is a subspace.

- Suppose: $\vec{b}_1 = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ then $b_1 + b_2 = (c_1+d_1)\vec{v}_1 + \cdots + (c_k+d_k)\vec{v}_k$
$\vec{b}_2 = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$ is in $\text{span}(\vec{v}_1, \ldots, \vec{v}_k)$
- Also $f\vec{b}_1 = (fc_1)\vec{v}_1 + (fc_2)\vec{v}_2 + \cdots + (fc_k)\vec{v}_k$ is also in $\text{span}(\vec{v}_1 \ldots \vec{v}_k)$

ex) $W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \text{ are real \#'s} \right\} \subseteq \mathbb{R}^3$

= "set of vectors $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ such that x, y are any real #"



Q: Is this a subspace?

1) Closed under addition:

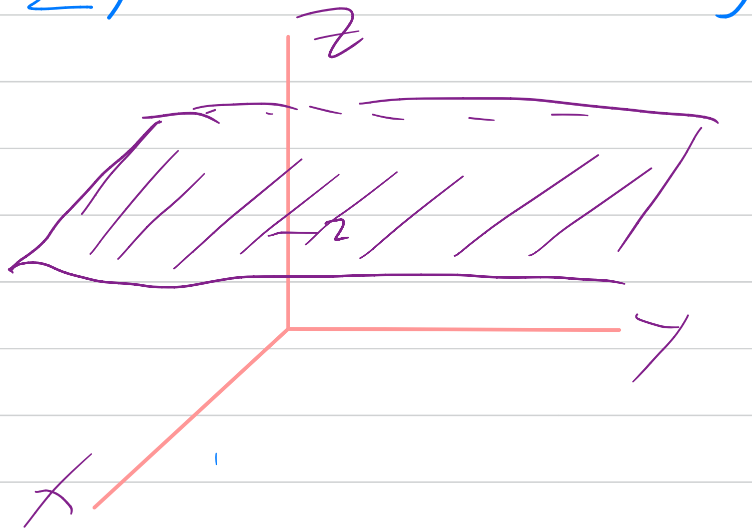
Check $\underbrace{\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}}_{\vec{v}_2} = \underbrace{\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{pmatrix}}_{\vec{v}_1+\vec{v}_2}$ is in W ✓

2) Closed under multiplication by a #.

Let $\vec{v} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ in W, c a real #.

Q: Is $c\vec{v}$ in W? Yes! $c\vec{v} = \begin{pmatrix} cx \\ cy \\ 0 \end{pmatrix}$

ex) $W = \left\{ \begin{pmatrix} x \\ y \\ 2 \end{pmatrix} : x, y \text{ are real \#'s} \right\}$



Q: Is W a subspace?

No! ex: $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is in W.

but $c\vec{v} = \begin{pmatrix} c \\ c \\ 2c \end{pmatrix}$ is not in W if $c \neq 1$.

ex) $W = \left\{ \begin{pmatrix} x \\ x+y \\ x-y \\ x^2 \end{pmatrix} : x, y \text{ are real \#'s} \right\} \subseteq \mathbb{R}^4$

<u>Is this a subspace?</u>

ex) $x=2$, $y=1$ $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}$, $x=3$, $y=0$ $\vec{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 9 \end{pmatrix}$

Note: $\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 5 \\ 6 \\ 4 \\ 13 \end{pmatrix}$ Q: Is $\vec{v}_1 + v_2$ in W?

It is not true that $x^2 + y^2 = (x+y)^2$ (in general)

ex) $W = \left\{ \begin{pmatrix} x \\ x+y \\ x-y \end{pmatrix} : x, y \text{ are real \#'s} \right\}$

Verify that W is a subspace!

- Closed under addition

Exercise (spelling ?)

Not Correct Way:

Show it for just a specific example

Basis - Combine the ideas of LI and spanning.

Def: Let $W \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ (could be $\mathbb{R}^n$ itself)
Then the vectors $\vec{v}_1 \dots \vec{v}_k$ of W are called basis for W if

1) the vectors are LI
2) if they span W.

Q: What's the meaning of this?

- 2 $\Rightarrow$ Every vector in W is a LC of $\vec{v}_1, \dots, \vec{v}_k$
- 1 $\Rightarrow$ The LC for every vector in W is unique!

ex: $W = \mathbb{R}^n$. Suppose $\vec{v}_1 \dots \vec{v}_k$ is a basis for $\mathbb{R}^n$

- then since $\vec{v}_1 \dots \vec{v}_n$ span $\mathbb{R}^n$, we must have $k \geq n$

- Also since $\vec{v}_1 \dots \vec{v}_n$ are LI we must have $k \leq n$

$\boxed{\Rightarrow \text{thus any basis for } \mathbb{R}^n \text{ must have } n \text{ vectors.}}$

Also note: As mentioned before, if $\vec{v}_1 \dots \vec{v}_n$ is a basis for $\mathbb{R}^n$ the linear system

$$\left(\begin{array}{ccc|c} \vec{v}_1 & \dots & \vec{v}_n & \vec{b} \\ \downarrow & & \downarrow & \downarrow \end{array} \right)$$

has a unique solution for any vectors $\vec{b}$

Thm: Every subspace $W \subseteq \mathbb{R}^n$ has a basis!

ex: Are the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ a basis for $\mathbb{R}^3$?

What to do: Check if pivot in every row + every collumn.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

ex) Consider the subspace $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 + x_3 = 0 \right\}$

(hyperplane)

Are the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ a basis for W?

1) Check they are LI as "usual"

SKIP

2) Now we show that any vector in W can be expressed as a linear combo of these two vectors.

SKIP